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ON STABILITY OF THE PLANE FLAME FRONT*

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The hydrodynamic stability of a plane flame front relative to two-dimensional perturbations in an incompressible fluid is considered. The effect of the flame front curvature and pressure perturbations ahead of it on the perturbed velocity of front propagation is taken into account. The possibility of stable combustion relative to long-wave perturbations and the existence of damped and intensified wave solutions and, also, of waves travelling along the flame front, depending on the model determining parameters, is disclosed.

Investigation of the plane flame front stability relative to two-dimensional perturbations in an incompressible fluid was initiated by Landau /1/, where its absolute instability was demonstrated on the assumption of constant flame velocity. The effect of front curvature on the flame propagation velocity was taken into account by Markstein /2/, which made it possible to show stability of combustion relative to short-wave perturbations and its instability relative to long wave ones. The question of existence of waves travelling along the flame front surface was investigated in /3/ in the case when the ratio of densities of reaction products and of the original mixture approaches zero. The combustion front stability relative to two-dimensional compressible perturbations was considered in /4,5/, and in /6/ stability was investigated relative to one-dimensional compressible perturbations in the piston-flame-shock wave system.

Let us consider the process of slow burning of a combustible mixture of gases. We assume, as in /1/ that the flame front is an infinitely thin gasdynamic discontinuity travelling at a specified constant velocity U_0 generally dependent on the composition and thermodynamic state of the combustible mixture. We shall use a system of coordinates attached to the flame front surface, with the x-axis directed along the velocity component normal to the discontinuity surface. We denote pressure by p_1, density by ρ_1, the normal and tangential velocity components ahead of the flame front by u_1, v_1, respectively, and by p_2, ρ_2, u_2, v_2 the respective parameters of the reaction products. For the model of combustion in incompressible fluid it is sufficient to specify the normal flame velocity U_0 and the thermal expansion coefficient α which is equal to the ratio of densities of products of combustion and of the original mixture.

Using the laws of mass and momentum conservation we obtain the relation between the gasdynamic parameters of combustible mixture and of reaction products at the discontinuity surface

$$u_1 = U_0,\ u_2 = U_0/\alpha,\ v_1 = v_2 = 0, \qquad \rho_2 = \alpha\rho_1, \quad p_2 = p_1 - \frac{1-\alpha}{\alpha}\rho_1 U_0^2 \tag{1}$$

Retaining the model of incompressible fluid, we investigate the flame front stability relative to plane two-dimensional periodic perturbations that are potential ahead of the front and turbulent behind it /1/. Denoting perturbations by a prime, we have at $x \leqslant 0$

$$\begin{aligned} u_1'/U_0 &= A \exp(kx + iky - i\omega t) \\ v_1'/U_0 &= iA \exp(kx + iky - i\omega t) \\ \frac{p_1'}{\rho_1 U_0^2} &= -(1+\Omega) A \exp(kx + iky - i\omega t) \end{aligned} \tag{2}$$

and at $x \geqslant 0$

$$\begin{aligned} u_2'/U_0 &= B\exp(-kx + iky - i\omega t) + C\exp\left(\frac{i\omega}{u_2}x + iky - i\omega t\right) \\ v_2'/U_0 &= -iB\exp(-kx + iky - i\omega t) - i\alpha\Omega C\exp\left(\frac{i\omega}{u_2}x + iky - i\omega t\right) \\ \frac{p_2'}{\rho_1 U_0^2} &= (\alpha\Omega - 1) B\exp(-kx + iky - i\omega t) \end{aligned} \tag{3}$$

where k is the wave number and Ω the dimensionless frequency.

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$$k = \frac{2\pi}{\lambda}, \quad \Omega = -\frac{i\omega}{kU_0} \tag{4}$$

As known in acoustics /7/, the considered here perturbations correspond for real ω to acoustic perturbations of a frequency lower than the critical $\omega_* = ka\sqrt{1-M^2}$, where a is the speed of sound and M is the Mach number.

We take the discontinuity surface perturbation in the form of a wave periodic with respect to y and t

$$x = \zeta(y, t), \quad \zeta(y, t) = \frac{D}{k}\exp(iky - i\omega t) \tag{5}$$

We assume the perturbed flame velocity to be

$$U = U_0\left(1 + \frac{\mu}{R} + \beta\frac{p_{10}'}{p_1}\right) \tag{6}$$

where μ and β are constants experimentally determined for a specific gas mixture and given thermodynamic conditions, p_{10}' is the perturbed pressure at the combustion front at $x = 0$, and R is the radius of the perturbed flame front curvature. In linear approximation

$$\frac{1}{R} = -\frac{\partial^2\zeta}{\partial y^2} = Dk\exp(iky - i\omega t) \tag{7}$$

Formula (6) takes into account the dependence of the combustion rate not only on flame surface perturbations, as in /2/, but, also, on perturbations of gasdynamic parameters ahead of the front (in the considered here case of incompressible fluid: on pressure perturbations). For $\beta = 0$ the proposed model becomes that of Markstein, hence we assume, as in /2/, that $\mu = \mu_0 L$, where $L = \varkappa/U_0$ is some characteristic dimension proportional to the combustion zone thickness and $\varkappa$ is the coefficient of thermal diffusivity. Value of the dimensionless constant μ_0 used by various authors differ, thus in /2/ $\mu_0 \sim 1$ and in /8,9/ $\mu_0 \sim 10 \div 20$.

Using (2) and (5)—(7) we obtain for flame velocity perturbation an expression of the form

$$\frac{U'}{U_0} = [\mu_1 D - \beta_1(1+\Omega)A]\exp(iky - i\omega t)$$
$$\mu_1 = \mu k = 2\pi\mu_0 L/\lambda, \quad \beta_1 = \beta\rho_1 U_0^2/p_1 \tag{8}$$

Linearizing the perturbation relations (1) at the flame front, we have at $x = 0$ the following boundary conditions:

$$u_1' - \frac{\partial\zeta}{\partial t} - U' = 0, \quad p_2' - p_1' + 2\frac{1-\alpha}{\alpha}\rho_1 U_0\left(u_1' - \frac{\partial\zeta}{\partial t}\right) = 0$$
$$u_1' - \alpha u_2' - (1-\alpha)\frac{\partial\zeta}{\partial t} = 0, \quad v_1' - v_2' - \frac{1-\alpha}{\alpha}U_0\frac{\partial\zeta}{\partial y} = 0 \tag{9}$$

Substituting perturbations (2), (3), (5), and (8) into system (9), we obtain a homogeneous linear system of algebraic equations in A, B, C, D. Equating to zero the determinant of this system, we obtain for Ω the cubic equation

$$a_0\Omega^3 + a_1\Omega^2 + a_2\Omega + a_3 = 0 \tag{10}$$
$$a_0 = (1-\alpha)\beta_1, \quad a_1 = \frac{(1-\alpha)(2+\alpha)}{\alpha}\beta_1 - (1+\alpha)$$
$$a_2 = 3\frac{1-\alpha}{\alpha}\beta_1 - 2(1+\mu_1), \quad a_3 = \frac{1-\alpha}{\alpha}(1+\beta_1) - \frac{2}{\alpha}\mu_1$$

The Routh—Hurwitz criterion /10/ shows that all roots of Eq.(10) have negative real parts then and only then when the system of inequalities

$$\beta_1\left[\beta_1 - \frac{\alpha(1+\alpha)}{(1-\alpha)(2+\alpha)}\right] > 0$$
$$\left(\beta_1 - \frac{\alpha}{1-\alpha}\right)\left[\frac{(1-\alpha)(3+\alpha)}{\alpha(1+\alpha)}\beta_1 - \mu_1 - 1\right] > 0$$
$$\beta_1\left[\frac{1-\alpha}{2}(1+\beta_1) - \mu_1\right] > 0$$

is satisfied.

The solution of this system is diagrammatically represented in Fig.1, by the unshaded regions in the plane (β_1, μ_1), where the straight line *1*

$$\mu_1 = \frac{(1-\alpha)(3+\alpha)}{\alpha(1+\alpha)}\beta_1 - 1$$

and straight line *2*

$$\mu_1 = \frac{1-\alpha}{2}(1+\beta_1)$$

Thus regions

$$\beta_1 < 0, \quad \mu_1 > \frac{1-\alpha}{2}(1+\beta_1) \tag{11}$$

$$\beta_1 > \frac{\alpha}{1-\alpha}, \quad \mu_1 < \frac{1-\alpha}{2}(1+\beta_1) \tag{12}$$

correspond in the considered here model to stable combustion.

In the shaded region of variation of parameters β_1, μ_1 there exist Ω with $\mathrm{Re}\,\Omega > 0$, while at the boundary neutrally stable ($\mathrm{Re}\,\Omega = 0$), as well as damped solutions are possible.

Let us consider in the plane (β_1, μ_1) the points and straight lines that correspond to known models. Thus the coordinate origin $\beta_1 = \mu_1 = 0$ which lies inside the determined above instability region (Fig.1) corresponds to Landau's model with constant flame propagation velocity $U = U_0$. When $\beta_1 = \mu_1 = 0$ the cubic equation (10) reduces to the quadratic one whose solution is /1/

$$\Omega_0 = \frac{-\alpha \pm (\alpha + \alpha^2 - \alpha^3)^{1/2}}{\alpha(1+\alpha)}$$

Both Landau's solutions are real with one of them for $\alpha < 1$ is positive. Length of the perturbation wave for which this model of plane combustion front is applicable must exceed the width of the combustion zone, hence the flame in Landau's model is unstable relative to perturbations in which the wave length $\lambda > L$.

The straight line $\beta_1 = 0$ corresponds to the Markstein's model. In Fig.1 the part of axis μ_1 lying above the point at ordinate $(1-\alpha)/2$ coincides with the stability limit, and its remaining part lies inside the instability region. When $\beta_1 = 0$ Eq.(10) has two roots /2/

$$\Omega_m = \frac{-\alpha(1+\mu_1) \pm (\alpha + \alpha^2 - \alpha^3 + \alpha^2\mu_1^2 - 2\alpha\mu_1)^{1/2}}{\alpha(1+\alpha)} \tag{13}$$

Hence when $\mu_1 > (1-\alpha)/2$, $\mathrm{Re}\,\Omega_m < 0$, and consequently combustion in Markstein's model is stable relative to short-wave perturbations whose wave length $\lambda < \lambda_m$ an unstable relative to perturbations of wave length $\lambda > \lambda_m$, where $\lambda_m = 4\pi\mu_0 L/(1-\alpha)$. Allowance for the effect of curvature on flame velocity results, as previously, in instability relative to long-wave perturbations (perturbations whose wave length considerably exceeds the flame front thickness).

In the model in /6/, unlike in those of Landau and Markstein, we obtain from conditions (12) for $\beta_1 > \alpha/(1-\alpha)$ stability of the flame front with perturbations to long-wave $\lambda > \lambda_*$ and to short-wave perturbations with $\lambda < \lambda_*$, where

$$\lambda_* = \frac{4\pi\mu_0}{(1-\alpha)(1+\beta_1)}L$$

When $\beta_1 = \alpha/(1-\alpha)$ the solution of Eq.(10) is of the form

$$\Omega_1 = -\frac{1}{\alpha}, \quad \Omega_{2,3} = \pm i\left(\frac{1-2\mu_1}{\alpha}\right)^{1/2}, \quad \mu_1 < \frac{1}{2}$$

$$\Omega_1 = -\frac{1}{\alpha}, \quad \Omega_{2,3} = \pm\left(\frac{2\mu_1 - 1}{\alpha}\right)^{1/2}, \quad \mu_1 > \frac{1}{2}$$

At stability limit $(\beta_1 = \alpha/(1-\alpha), 0 \leqslant \mu_1 \leqslant 1/2$ we have, besides the damped solution, neutrally stable travelling waves whose propagation velocity along the flame front is $U_0\sqrt{(1-2\mu_1)/\alpha}$. When $\beta_1 = \alpha/(1-\alpha)$, the flame is unstable relative to short-wave perturbations $(\lambda < 4\pi\mu_0 L)$.

Further analysis of solutions of the cubic equation (10) is linked with the determination in the plane (β_1, μ_1) of domains of complex roots of this equation. When $\mathrm{Im}\,\Omega \neq 0$ we call solutions of Eq.(10), wave solutions. The perturbation propagation velocity along the combustion front is then $U_0\,\mathrm{Im}\,\Omega$. Depending on the sign of $\mathrm{Re}\,\Omega$ wave solutions are either damped ($\mathrm{Re}\,\Omega < 0$) or increasing ($\mathrm{Re}\,\Omega > 0$). We call perturbations whose $\mathrm{Re}\,\Omega = 0$, travelling waves.

The cubic equation (10) may have either three different real roots when its discriminant is negative, or three real roots, two of which are the same when $Q = 0$, or, when $Q > 0$, where

$$Q = q^2 + p^3, \quad p = \frac{3a_0a_2 - a_1^2}{9a_0^2}, \quad q = \frac{a_1^3}{27a_0^3} - \frac{a_1a_2}{6a_0^2} + \frac{a_3}{2a_0} \tag{14}$$

one real and two complex conjugate roots.

The limit of wave solution existence is determined by the condition

$$Q = 0 \tag{15}$$

Substituting in (15) expressions (14) for coefficients p, q, and (10) for a_i, we obtain for μ_1 the cubic equation

$$m_0\mu_1^3 + m_1\mu_1^2 + m_2\mu_1 + m_3 = 0$$
$$m_0 = p_1^3, \quad m_1 = 3p_0p_1^2 + q_1^2$$
$$m_2 = 3p_0^2p_1 + 2q_0q_1, \quad m_3 = p_0^3 + q_0^2$$

To eliminate μ_1 in coefficients p, q, and a_i we introduce the following notation:

$$a_2 = a_{20} + a_{21}\mu_1, \quad a_{20} = 3\frac{1-\alpha}{\alpha}\beta_1 - 2, \quad a_{21} = -2$$
$$a_3 = a_{30} + a_{31}\mu_1, \quad a_{30} = \frac{1-\alpha}{\alpha}(1+\beta_1), \quad a_{31} = -\frac{2}{\alpha}$$
$$p = p_0 + p_1\mu_1, \quad p_0 = \frac{3a_0a_{20} - a_1^2}{9a_0^2}, \quad p_1 = \frac{a_{21}}{3a_0^2}$$
$$q = q_0 + q_1\mu_1, \quad q_0 = \frac{a_1^3}{27a_0^3} - \frac{a_1a_{20}}{6a_0^2} + \frac{a_{30}}{2a_0},$$
$$q_1 = -\frac{a_1a_{21}}{6a_0^2} + \frac{a_{31}}{2a_0}$$

The limit of existence of wave solutions of Eq.(10) for $\alpha = 0.2$ is shown in Fig.2. The two upper branches of curve $Q = 0$ are denoted there by $\mu_{11}(\beta_1)$ and $\mu_{12}(\beta_1)$, and the lower curve by $\mu_{13}(\beta_1)$. The straight lines *1* and *2* are the same as in Fig.1. The relative position of domains of instability and of existence of wave solutions shown in Fig.2 is valid for any α. Thus curve $\mu_{13}(\beta_1)$ is tangent to straight lines *1* and *2*, with the point of tangency with line *2* at $(\alpha/(1-\alpha); 0.5)$, as can be verified by substitution.

We denote by β_1^0 the value of β_1 for which curve $\mu_{13}(\beta_1)$ intersects axis $\mu_1 = 0$. It will be seen that β_1^0 is close to

$$\beta_1 = \frac{\alpha(1+\alpha)}{(1-\alpha)(3+\alpha)}$$

at which straight line *1* intersects the β_1 axis (Fig.2).

The intersection points of curves $\mu_{11}(\beta_1)$ and $\mu_{12}(\beta_1)$ with axis $\beta_1 = 0$ are determined by the sign of the radicant in Markstein's solution (13). We find that when

$$\mu_{12}^0 \leqslant \mu_1 \leqslant \mu_{11}^0, \quad \mu_{1i}^0 = \frac{1-(-1)^i(1-\alpha-\alpha^2+\alpha^3)^{1/2}}{\alpha} \tag{16}$$

at the flame front surface damped wave solutions with

$$\operatorname{Re}\Omega_m = -\frac{1+\mu_1}{1+\alpha} < 0 \tag{17}$$

moving along the y-axis at velocity

$$V = U_0 \operatorname{Im}\Omega_m = U_0\frac{[1-\alpha-\alpha^2+\alpha^3-(\alpha\mu_1-1)^2]^{1/2}}{\alpha(1+\alpha)} \tag{18}$$

are possible.

The limit μ_{1i}^0 of wave solutions in Markstein's model is shown in Fig.3, as a function of α, and in Fig.4 is shown the dependence of the reduced value of wave propagation velocity $V/U_0 = \operatorname{Im}\Omega_m$ on μ_1 which is inversely proportional to wave length. Conditions (16) imply that in Markstein's model wave solutions are only possible in the case of fairly short-wave perturbations

$$2\pi\mu_0L/\mu_{11}^0 \leqslant \lambda \leqslant 2\pi\mu_0L/\mu_{12}^0$$

Consider the limit case of $\alpha \to 0$ when $\beta_1 = 0$. From condition (16) we obtain the wave solution existence limit

$$\mu_1 > {}^1/_2 \tag{19}$$

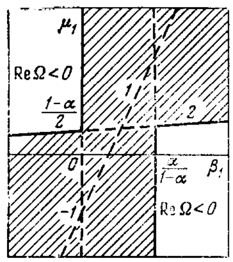

Fig.1

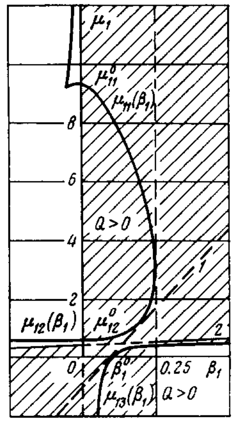

Fig.2

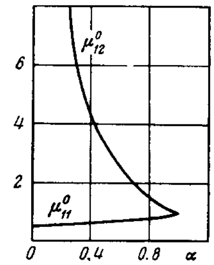

Fig.3

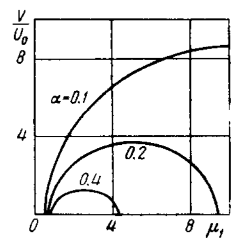

Fig.4

and from (17) and (18) the real part and the principal term of wave perturbation propagation velocity along the flame front

$$\mathrm{Re}\,\Omega_m = -(1+\mu_1) < 0, \quad V = U_0\alpha^{-1/2}(2\mu_1 - 1)^{1/2} \tag{20}$$

Formula (19) and the first of relations (20) were obtained in /3/ for extremely small α $(\alpha \to 0)$ when investigating waves travelling along the flame front $(\mathrm{Re}\,\Omega = 0)$. The second of relations (20) implies that wave perturbations in Markstein's model must be damped as $\alpha \to 0$.

Let us revert to the general case. The increase of perturbations depends on the index of exponent

$$-i\,(\mathrm{Im}\,\omega)t = \mu_1\,(\mathrm{Re}\,\Omega)U_0 t\,/\,(\mu_0 L)$$

because of this we introduce parameter $S = \mu_1\,\mathrm{Re}\,\Omega$ which defines the increase $(S > 0)$ or decrease $(S < 0)$ of perturbations during the time it takes the front to pass the distance proportional to the combustion zone thickness.

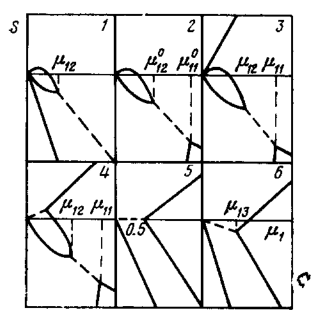

Fig.5

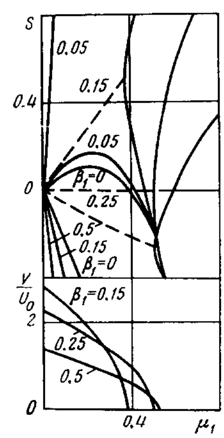

Fig.6

The dependence of parameter S on μ_1, which is inversely proportional to the wave length, for several values of parameter β_1 is shown diagrammatically in Fig.5, where solid lines relate to $\mathrm{Im}\,\Omega = 0$ and the dash ones to $\mathrm{Im}\,\Omega \neq 0$.

For $-1 < \beta_1 < 0$ (Fig.5,*1*) the solution in the domain of long waves is increasing, and in that of short waves we have damped wave solutions. When $\beta_1 = 0$ (Fig.5,*2*) there are only two Markstein's solutions. When $0 < \beta_1 \leqslant \beta_1{}^0$ (Fig.5,*3*) the considered here model is unstable relative to perturbations of any wave length. When $\beta_1 > \beta_1{}^0$ we have in the range of long waves

($\mu_1 < \mu_{13}$) wave solutions that are increasing for $\beta_1^0 < \beta_1 < \alpha / (1-\alpha)$, neutrally stable for $\beta_1 = \alpha / (1-\alpha)$, and damped when $\beta_1 > \alpha / (1-\alpha)$ (Fig.5, $4-6$).

When $\beta_1^0 < \beta_1 < \alpha / (1-\alpha)$ we have wave perturbations on the flame front surface which move along the y-axis and increase with time. Their nonlinear inter-action may result in some form characteristic of a standing wave which can be related to the cellular flame structure observed experimentally /2/.

Let us assume that the nonlinear analysis led to the determination of μ_1^* that corresponds to the obtained resultant perturbation, then such perturbation wave length or dimension of the cell is $\lambda^* = 2\pi\mu_0 L / \mu_1^*$. Let us obtain the following estimate when $\mu_1^* \approx 0.25$. Then with $\mu_0 \sim 1, L \sim 3 \cdot 10^{-2}$ cm we obtain $\lambda^* \sim 1$ cm, i.e. the cell dimension is close to the experimentally obtained /2,11/. Note that the cell dimension calculated earlier using Markstein's model with μ_1 which corresponds to the maximum of S (Fig.5,2) is of the same order, namely, $8\pi\mu_0 L / (1-\alpha)$ (with $\mu_1 = (1-\alpha)/4$).

When $\beta_1 = \alpha / (1-\alpha)$, travelling waves of length $\lambda > 4\pi\mu_0 L$ moving along the combustion front at velocity $V = U_0 [(1-2\mu_1)/\alpha]^{1/2}$ are generated on the flame front. When $\beta_1 > \alpha / (1-\alpha)$ (Fig.5, 6), we have results that conform to experimental data /12, 13/ as regards combustion front stability relative to long-wave perturbations and instability relative to short-wave ones.

The dependence of parameter S and of wave perturbation propagation velocity along the flame front on μ_1 is shown in Fig.6 for $\alpha = 0.2$ and several values of β_1.

Note that the establishement of flame front stability relative to long-wave perturbations and the determination of the domain of wave solution existence was made possible by taking into account the effect of pressure perturbations ahead of the flame front on the combustion rate.

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